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A Survey of Solved and Unsolved Problems on Superpositions of Functions

DAVID A. SPRECHER

Department of Mathematics, University of California, Santa Barbara, California 93106

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One tool in the search of new approximation schemes and computational algorithms for functions of several variables is the study of their representability in terms of prescribed superpositions of functions of fewer variables [21, 36]. We shall outline in this paper some ideas which originated with Hilbert, but were carried far beyond their original purpose by Kolmogorov and his school.

The history of these ideas is sketched in Section 1; some of the main results in the area of superpositions are outlined in Sections 2 and 3; Section 4 describes unsolved problems. The reader is also referred to the excellent papers of Arnol'd [4], Lorentz [22], Vituškin [39], and Vituškin–Henkin [40].

1. HILBERT'S PROBLEM

The idea of representing functions of n variables as superpositions of functions of $m < n$ variables for the purpose of studying the structure of function classes is due to Hilbert. He first drew attention to this circle of ideas in the thirteenth of his celebrated twenty three problems [15], and again twenty seven years later [16]. The thirteenth problem reads, in part, as follows: "... Likewise the general (polynomial) equations of the 5th and 6th degrees are solvable by suitable nomographic tables; for, by means of Tschirnhausen transformations, which require only extraction of roots, they can be reduced to a form where the coefficients depend upon two parameters only.

"Now it is probable that the root of the (polynomial) equation of the seventh degree is a function of its coefficients which does not belong to this class of functions capable of nomographic construction, i.e., that it cannot be constructed by a finite number of superpositions of functions of two arguments. In order to prove this, the proof would be necessary *that the equation of the seventh degree*

$$f^7 + xf^3 + yf^2 + zf + 1 = 0 \tag{1}$$

is not solvable with the help of any continuous functions of only two variables. I may be allowed to add that I have satisfied myself by a rigorous process that there exist analytic functions of three variables x, y, and z which cannot be obtained by a finite chain of [analytic] functions of only two arguments."

We note that this conjecture is algebraic in origin. It emerged out of the attempts to eliminate, by algebraic means, the largest possible number of coefficients from polynomial equations $\sum_{k=1}^{n} a_k x^k = 0$, thereby expressing their roots, regarded as functions of $n + 1$ coefficients, as functions of fewer coefficients [41]. Hilbert recognized the applicability of this idea to the more general problem alluded to above. While the conjecture itself was proved by Kolmogorov [19, 20] and Arnol'd [1] to be false, the fact must not be overlooked that the problem to which Hilbert addressed himself remains unsolved. Namely, *it is not known if Eq.* (1) *is solvable by finitely many superpositions of algebraic functions of two variables.* The word "analytic" which appears in brackets in the above quotation is missing in Hilbert's formulation, but there is no doubt that this was just an oversight on his part; it was known to him that *every* function of three variables is a superposition of finitely many functions of two variables [27]. Let us, in fact, outline a proof of the following fact:

THEOREM 1.1. *There is an analytic function of three variables which is not a finite superposition of analytic functions of two variables.*

Proof: For each integer $N \geqslant 1$ we define iteratively a superposition of order N of analytic functions of two variables by means of the following scheme:

$$\left.\begin{array}{lll}
Q^N = Q^N(p_1{}^1, p_2{}^1), & & \\
p_k{}^1 = p_k{}^1(p^2_{2k-1}, p^2_{2k}), & k = 1, 2, & \\
p_k{}^2 = p_k{}^2(p^3_{2k-1}, p^3_{2k}), & k = 1, 2, 3, 4 & \\
\quad\vdots & & \\
p_k{}^j = p_k{}^j(p^{j+1}_{2k-1}, p^{j+1}_{2k}), & k = 1, 2, \ldots, 2^j, & \\
\quad\vdots & & \\
p_k{}^N = p_k{}^N(x, y, z), & k = 1, 2, \ldots, 2^N. &
\end{array}\right\}
\begin{array}{l}
p_j{}^k \text{ are analytic functions of 2} \\
\text{variables; in particular, } p_k{}^N \text{ are} \\
\text{analytic functions of at most two} \\
\text{of the variables } x, y, z.
\end{array}$$

A simple count shows that Q^N, with the indicated insertions, is a superposition of $2^{N+1} - 1$ (nontrivial) analytic functions; we call Q^N *a superposition of order* N. Thus, a superposition of order 3 is

$$Q^3 = Q^3\{p_1^1[p_1^2(p_1^3, p_2^3), p_2^2(p_3^3, p_4^3)], p_2^1[p_3^2(p_5^3, p_6^3), p_4^2(p_7^3, p_8^3)]\}.$$

The following facts are easy to verify:

(a) If $\mathcal{S}^N$ stands for the set of superpositions of order N, then $\mathcal{S}^1 \subset \mathcal{S}^2 \subset \mathcal{S}^3 \subset \cdots$.

(b) There are only finitely many distinct superpositions of order N for each N. These are obtained by looking at the permutations of the space-variables x, y, and z in a fixed superposition of order N.

(c) For any finite superposition T of analytic functions of two variables there is a smallest integer N such that T is of order N.

The proof of the theorem is based on the following observation: Let T be an arbitrary superposition of order N of analytic functions of two variables, and look at all terms in T, after an appropriate manipulation, of degree $\leqslant m$ in each variable x, y, and z. The total number of independent coefficients corresponding to these terms does not exceed $N \cdot m^2$. On the other hand, in the expression

$$\sum_{i,j,k=1}^{\infty} a_{ijk}\, x^{\alpha_i} y^{\beta_j} z^{\gamma_k}, \tag{2}$$

in which α_i, β_j, and γ_k vary independently over the set of nonnegative integers, there are m^3 distinct terms of degree $\leqslant m$ in x, y, and z. The number of possible independent coefficients is therefore also m^3.

Now let N be fixed and suppose T equals the series (2). If $m > N$, then $m^3 > N \cdot m^2$, and it follows that the coefficients $a_{ijk} \geqslant 0$ of terms of degree $\leqslant m$ in x, y, and z must satisfy algebraic relations which depend on the particular superposition T. It is clear that coefficients a_{ijk} can be so selected that they do not satisfy any such algebraic relation, since there are only finitely many superpositions of order $\leqslant N$. Moreover, this can be done in such a way that coefficients a_{ijk} will decrease in a prescribed manner with increasing i, j, and k.

An inductive procedure can be developed for selecting coefficients a_{ijk} so that none of the required algebraic relations associated with superpositions of order 1, 2, 3 $\cdots$ is satisfied, and so that (2) will represent an analytic function.

2. Kolmogorov's Theorem

Designating now by E^n the n-fold cartesion product $E \times E \times \cdots \times E$ of the unit intervals $E = [0, 1]$, we let $\mathscr{C}(E^n)$ stand for the Banach space of real valued continuous functions defined on E^n, with the uniform norm.

In 1956 Kolmogorov obtained the very unexpected result that every function $f \in \mathscr{C}(E^n)$, $n \geqslant 4$, is a finite superposition of continuous functions of only 3 variables [19]. Specifically, he showed that each $f \in \mathscr{C}(E^n)$ can be represented as

$$f(x_1, \ldots, x_n) = \sum_{r=1}^{n} h^r[x_n, g_1{}^r(x_1, \ldots, x_{n-1}), g_2{}^r(x_1, x_2, \ldots, x_{n-1})], \tag{3}$$

where $h^r \in \mathscr{C}(E^3)$ and $g_j{}^r \in \mathscr{C}(E^{n-1})$. The above statement follows when this formula is applied repeatedly to the inner functions $g_j{}^r$. In this theorem, $(g_1{}^r, g_2{}^r)$ represents a point in the universal tree (which can be realized as a continuum in E^2). Consequently, the domain of the functions h^r is the cartesian product of a tree and the interval E. The proof of this theorem is rather difficult. It was followed in 1957 by a theorem of Arnol'd [1] that every $f \in \mathscr{C}(E^3)$ can be represented in the form

$$f(x_1, x_2, x_3) = \sum_{i,j=1}^{3} h_{ij}[\varphi_{ij}(x_1, x_2), x_3], \tag{4}$$

where the h_{ij} and φ_{ij} are continuous functions of two variables. This theorem disproved the conjecture in Hilbert's thirteenth problem. A detailed proof of this theorem can be found in [2] (see also [3] in this connection).

Analyzing the constructions in [1] and [19], Kolmogorov realized that the use of trees could be avoided and a much stronger result proved. It is the following remarkable superposition theorem [20]:

Theorem 2.1. *For each integer $n \geqslant 2$ there exist monotonic increasing functions $\psi^{pk} \in \mathscr{C}(E)$ with the property that every function $f \in \mathscr{C}(E^n)$ has a representation*

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{2n+1} g_k\left(\sum_{p=1}^{n} \psi^{pk}(x_p)\right), \tag{5}$$

where also the functions g_k are continuous.

Kolmogorov based this theorem on three lemmas which he stated without proof. Proofs of these, as well as alternative proofs of the theorem, were subsequently given by the author [28], Kim [18], Lorentz [22, 23], Tihomirov [32], and others.

It was first observed by Lorentz [22] that the functions g_k can be replaced by a single function g. The author has shown that the theorem can be proved with constant multiples of a single function ψ and translations [28]. Specifically, for each integer $n \geqslant 2$ there is a monotonic increasing function $\psi \in \mathscr{C}(E)$ and constants $\epsilon > 0$ and $\lambda > 0$ with the property that each $f \in \mathscr{C}(E^n)$ has a representation of the form

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{2n+1} g\left[\sum_{p=1}^{n} \lambda^p \psi(x_p + \epsilon k) + k\right]. \tag{6}$$

Instead of powers of a single constant λ, one can use constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ which are linearly independent over the field of rationals. This follows at once from the constructions in [28]. We have also shown that the fixed inner functions must depend on k in a nontrivial way [29]. Specifically, let $\psi_p \in \mathscr{C}(E)$, $1 \leqslant p \leqslant n$, be artibrary functions, and let N be a positive integer. Then for any polynomial $v(x_2, \ldots, x_n)$,

$$x_1^{N+1} + v(x_2, \ldots, x_n) \neq \sum_{k=1}^{N} g\left[\sum_{p=1}^{n} \alpha_{pk} \psi_p(x_p) + \beta_k\right], \tag{7}$$

where g is continuous, and α_{pk} and β_k are arbitrary constants.

The most significant improvement in Kolmogorov's theorem, however, is due to Fridman [9]. He succeeded in showing that the functions ψ^{pk} in (5) can be constructed to belong to class Lip(1).[1] Using Fridman's construction we were able to show that also the single function ψ in (6) can be taken to belong to the class Lip(1) [31]. Earlier efforts have shown that the functions ψ^{pk} obtained by variations of Kolmogorov's construction can belong to classes Lip(α) for $0 < \alpha < 1$, excluding only (and specifically) the case $\alpha = 1$ [10, 22, 28]). Fridman's construction differs from that of Kolmogorov in an essential way, and his result was rather surprising. It is to be hoped that this improvement will make Kolmogorov's theorem more accessible to applications. It may also admit the use of distributions in further investigations of this theorem. For example, we used this technique in [29] to prove the assertion (7).

3. Superpositions with Smooth Functions

It is an elementary observation that smooth functions can be composed of nonsmooth ones, but nonsmooth functions cannot, in general, be composed of smooth ones. Because the space-variables $x_1, \ldots, x_n$ are independent, it

[1] $\psi(t)$ belongs to class Lip(α), $0 < \alpha \leqslant 1$, if there is a constant A such that $|\psi(t_1) - \psi(t_2)| \leqslant A |t_1 - t_2|^\alpha$ for all points t_1 and t_2 in its domain.

was expected that the possible nonsmoothness of members of $\mathscr{C}(E^n)$ is related to n: the larger n, the less smooth are the worst members of $\mathscr{C}(E^n)$. Hilbert thought to exploit this idea by using the smallest number of variables in the representations of a continuous function in terms of superpositions as a classification index [16]. Kolmogorov's theorem 2.1 shows that all continuous functions have index $\chi = 1$, and hence the idea of using the number of variables as a classification index has failed. Vituškin has discovered, however, that if instead of $\mathscr{C}(E^n)$ we consider $\mathscr{C}^{(p)}(E^n)$, the space of functions of n variables all of whose partial derivatives of orders $\leqslant p$ exist and are continuous, then the index $\chi = n/p$ works $(p \geqslant 1)$ [33]. This is one of the deepest results in the area of superpositions, and it can be stated as follows:

THEOREM 3.1. *Not all functions of index $\chi = n/p$ can be realized as a superposition of functions of index $\chi_0 = (n_0/p_0) < \chi$.*

Vituškin's proof uses the concept of multidimensional variation developed by him. Another proof, using ϵ-capacity, is contained in a paper of Kolmogorov and Tihomirov [21]. The theorem demonstrates the inevitable decrease in smoothness in representations by superpositions as the number of variables decreases.

We now list some special results involving superpositions with smooth functions. The first was proved by Ostrowski [26]:

THEOREM 3.2. *The analytic function*

$$\zeta(x, y) = \sum_{k=1}^{\infty} x^k/k^y$$

is not a finite superposition of infinitely differentiable functions of one variable, and algebraic functions of any number of variables.

Closely related to Kolmogorov's theorem is the following result of Vituškin [37, 38]:

THEOREM 3.3. *Let $\varphi_k(x_1, x_2)$, $1 \leqslant k \leqslant N$, be arbitrary functions of $\mathscr{C}(E^2)$, and let $\psi_k(x_1, x_2)$, $1 \leqslant k \leqslant N$, be continuously differentiable. Then there is an analytic function of two variables which is not representable in the form*

$$\sum_{k=1}^{N} \varphi_k(x_1, x_2) \cdot g_k[\psi_k(x_1, x_2)] \tag{8}$$

with continuous functions g_k.

Extending this result, Henkin [10] has shown that the set of superpositions (8) is closed and nowhere dense in $\mathscr{C}(E^2)$. At the same time, he constructed a polynomial $(x_1 + \nu x_2)^\mu$ which cannot be written in the form (8).

While the proofs of Vituškin and Henkin can be generalized to encompass the superpositions

$$\sum_{k=1}^{N} \varphi_k(x_1, \ldots, x_n)\, g_k[\psi_k(x_1, \ldots, x_n)],$$

they do not seem to apply to superpositions

$$\sum_{k=1}^{N} \varphi_k(x_1, \ldots, x_n)\, g_k[\psi_{k1}(x_1, \ldots, x_n), \ldots, \psi_{km}(x_1, \ldots, x_n)], \tag{9}$$

where $1 < m < n$, the functions φ_k are continuous, and the functions Ψ_{kj} are continuously differentiable. We thus have the following

3.4. *Problem*

Is there an analytic function of $n \geqslant 3$ variables which cannot be represented as a superposition of the form (9) with continuous functions φ_k and g_k, and with continuously differentiable functions ψ_{kj} ?

We close this section with two conjectures of Kolmogorov.[2]

3.5. *Conjectures*

There exist analytic functions of three variables which are not representable as finite superpositions of continuously differentiable functions of two variables; there exist analytic functions of two variables which are not representable as finite superpositions of continuously differentiable functions of one variable.

A proof of a special case of the second conjecture is contained in Vituškin's proof of theorem 3.3. It should be noted that, for $n = 3$, Problem 3.4 is a special case of the first conjecture in 3.5.

4. Problems

In the last section we stated a problem and two conjectures dealing with more general superpositions. Returning to the basic form in which Kolmogorov's theorem is stated, we consider here problems connected with superpositions of the form

$$\sum_{k=1}^{N} g[\psi_k(x_1, \ldots, x_n)]. \tag{10}$$

[2] Communicated privately to the author.

4.1. *Uniqueness*

Are there functions $\psi_k \in \mathscr{C}(E^n)$, $k = 1, 2, \ldots, N$, *with the property that each function* $f \in \mathscr{C}(E^n)$ *has a unique representation of the form* (10)? Clearly, this is equivalent to asking if zero has a unique representation in the form (10). An affirmative answer to this question would give, as a corollary, a new proof for the affirmative answer to the following problem of Banach [6]:

Are the spaces $\mathscr{C}(E^n)$ *and* $\mathscr{C}(E)$ *isomorphic* (linearly homeomorphic)? The solution of this problem is contained in a paper of Miljutin [24] published in 1966. The result, however, is already contained in his doctoral dissertation of 1952.

This line of investigation goes far beyond the study of superpositions. We mention here one result of Henkin [11] which complements the result of Miljutin:

Let $\mathscr{C}^{(s)}(E^n)$ stand for the Banach space of continuous functions defined on E^n and having continuous partial derivatives of orders $\leqslant s$, the norm being the usual one. If $p \geqslant 0$, $s \geqslant 1$, and $n \geqslant 2$, then there is no linear homeomorphism between the spaces $\mathscr{C}^{(s)}(E^n)$ and $\mathscr{C}^{(p)}(E)$.

Other results along these lines can be found in the work of Henkin [12–14], Kadec [17], and others.

4.2. *Convergence*

Given continuous functions $g_{kj}(t)$, $\psi_{kj}(x_1, \ldots, x_n)$ such that the uniform limit

$$\lim_{j\to\infty} \sum_{k=1}^{N} g_{kj}[\psi_{kj}(x_1, \ldots, x_n)] = f(x_1, \ldots, x_n) \tag{11}$$

exists, there is no guarantee that this limit is of the form (10). The problem, then, is to find necessary and sufficient conditions characterizing the class of sequences whose uniform limit (11) exists and is of the form (10). To make this problem meaningful, some restrictions, must be imposed on the functions ψ_{kj}.

Even when $N = 1$, the answer is not known, a case in point being the uniform limit ($0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$)

$$xy = \lim_{j\to\infty} \exp\Big[\ln\Big(x + \frac{1}{j}\Big) + \ln\Big(y + \frac{1}{j}\Big)\Big].$$

The functions $g_j(t) \equiv \exp(t)$ and $\varphi_j(t) \equiv \ln(t + 1/j)$ are strictly monotonic, and $f(x, y) \equiv xy$ is strictly monotonic in each variable, except when $x = 0$ or $y = 0$. Yet, the limit of the sequence $g_j[\varphi_j(x) + \varphi_j(y)]$ is not of the form $g[\varphi(x) + \psi(y)]$ [5, 30].

A partial answer to the question of convergence of sequences of the form $g_j[\varphi_j(x) + \psi_j(y)]$ was given by Vainštein and Kraines [33], who showed that the limit is of the form $g[\varphi(x) + \psi(y)]$ if it is strictly monotonic in each variable. According to our note [30], this condition is not necessary.

4.3. *Minimal Number of Summands*

Bassalygo [7] has shown that given any functions $\psi_k \in \mathscr{C}(E^2)$, $k = 1, 2, 3$, there is a function $f \in \mathscr{C}(E^2)$ not representable in the form $\sum_{k=1}^{3} g_k[\psi_k(x_1, x_2)]$. Returning now to the specific representation in Kolmogorov's theorem 2.1, *it is not known if* $2n + 1$ *is the smallest number of summands when* $n > 2$.

The case $n = 2$ has been settled by Doss [8] who has shown that formula (5) with $2n + 1$ replaced by $2n$ is not true for all functions $f \in \mathscr{C}(E^2)$ when the functions ψ^{pq} are monotonic.

It is interesting to note that the argument of Doss depends only on the fact that each function $\psi^{1k}(x_1) + \psi^{2k}(x_2)$, by virtue of the stipulated monotonicity, has level sets which intersect given level sets of the remaining functions in a prescribed manner. This fact, however, plays no explicit role in Kolmogorov's proof of theorem 2.1. Although Doss's construction becomes complicated already for four summands, it might be interesting to discover why the argument breaks down when five summands are used. In fact, we know from Vituškin's Theorem 3.3. that Doss's argument would *not* break down even with more than five summands when the functions ψ^{pk} are assumed to be continuously differentiable.

4.4. *Characterization of the Function* ψ_k

Disregarding the particular form of the right hand side of (5), Kolmogorov's Theorem 2.1 can be stated as follows:

THEOREM. A. *Let* $\{S_{i1}^k, \ldots, S_{ik_i}^k\}$, $k = 1, 2, \ldots, 2n + 1$, $i = 1, 2, 3, \ldots$, *be families of closed n-cubes with the following properties*:

(a) $S_{ir}^k \cap S_{iq}^k = \phi$, *whenever* $r \neq q$;

(b) *diameter* $(S_{ir}^k) \to 0$ *as* $i \to \infty$;

(c) *for each value of* i, *every point of* E^n *belongs to at least* $n + 1$ *n-cubes* S_{ir}^k.

B. *Let* $\psi_k \in \mathscr{C}(E^n)$, $k = 1, 2, \ldots, 2n + 1$, *be functions endowed with the property that* $\psi_k(S_{ir}^k) \cap \psi_k(S_{iq}^k) = \phi$ *whenever* $r \neq q$, *for all* i *and* k, $\psi_k(S_{ir}^k)$ *designating the image of* S_{ir}^k *under* ψ_k.
Then each function $f \in \mathscr{C}(E^n)$ *can be represented in the form*

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{2n+1} g_k[\psi_k(x_1, \ldots, x_n)], \tag{12}$$

where the functions g_k *are continuous.*

In the various proofs of Kolmogorov's theorem, the functions ψ_k are always defined by their separation of cubes as in the formulation above. It would be desirable to have a direct (analytical) characterization of those functions ψ_k which admit a Kolmogorov type theorem.

From the above formulation of the theorem it is quite apparent that each function ψ_k is one-to-one on a large subset of E^n. In fact, the following can be deduced:

C. *There are subsets A_k of E^n with the properties*

(d) *for each k, ψ_k is one-to-one on A_k ;*

(e) *each point of E^n belongs to at least $n + 1$ sets A_k . We now pose the following specific questions*:

(i) Does the conclusion of the above theorem hold if condition A and B are replaced by C?

(ii) Is condition C necessary for a Kolmogorov type theorem?

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